General metrics and discrete operators on multigraphs

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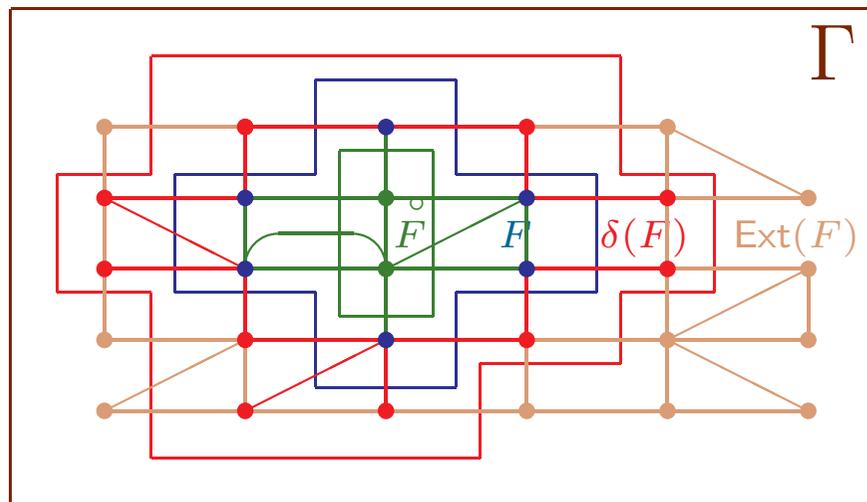
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Discrete Manifolds

$\triangleright$ $\Gamma = (V, E, \theta)$ Finite Multigraph

$\triangleright$ $E_{xy} = \{e : \theta(e) = \{x, y\}\}$, $E_x = \bigcup_{y \sim x} E_{xy}$,

  $|E_x| = k(x) \equiv$ Degree of x

$\triangleright$ $\delta(F)$ Vertex Boundary $\triangleright$ $F = \mathring{F} \cup \delta(F^c)$

$\triangleright$ $\partial(F)$ Edge Boundary $\triangleright$ $\bar{F} = F \cup \delta(F)$ Clousure

Fields on a Discrete Manifold
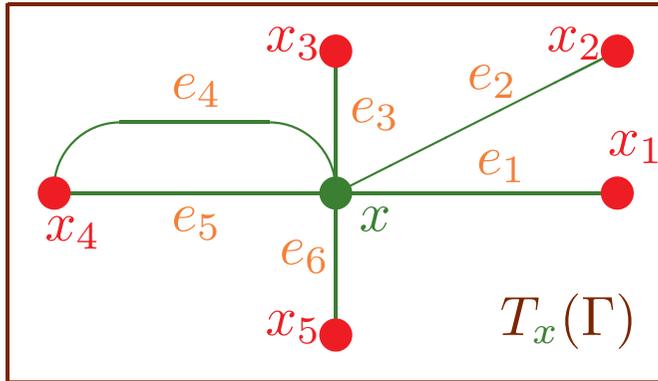
▷ Tangent space at $x : T_x(\Gamma)$

▷ Base $\{e_j\}_{j=1}^{k(x)}$

▷ Vectorial field on Γ: $\mathsf{f}(x) \in T_x(\Gamma)$
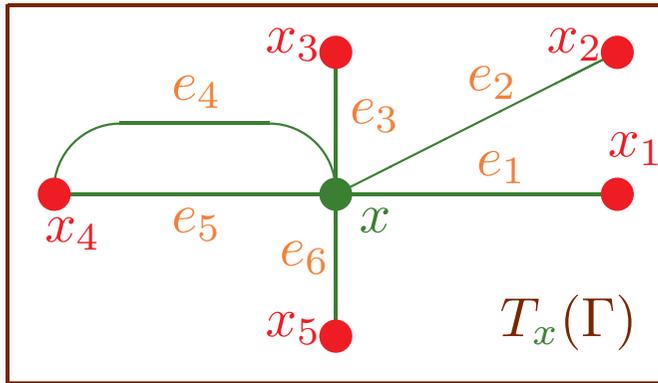
$$\mathsf{f}(x) = \sum_{j=1}^{k(x)} f(x, e_j)\, e_j = \sum_{e \in E_x} f(x, e)\, e$$

⋆ Basic decomposition: $\mathcal{X}(\Gamma) = \mathcal{X}^s(\Gamma) \oplus \mathcal{X}^a(\Gamma)$

$$\mathsf{f}^s(x) = \tfrac{1}{2} \sum_{y \sim x} \sum_{e \in E_{xy}} (f(x, e) + f(y, e))\, e,$$

$$\mathsf{f}^a(x) = \tfrac{1}{2} \sum_{y \sim x} \sum_{e \in E_{xy}} (f(x, e) - f(y, e))\, e$$

Fields on a Discrete Manifold

$\triangleright$ Field of bilinear forms on Γ: $\mathsf{B}(x) \in \mathcal{B}_x(\Gamma)$

$\triangleright$ Matrix Field on Γ:

$$\mathsf{M}(x) \in \mathcal{M}_{k(x)}(\mathbb{R}) \Longrightarrow m(x, e, e'), \quad e, e' \in E_x$$

$\triangleright$ Order of an operator

$$\mathsf{F} \colon \mathcal{V} \longrightarrow \mathcal{X}(\Gamma), \quad \mathsf{H} \colon \mathcal{V} \longrightarrow \mathcal{C}(V), \quad \mathcal{V} \subset \mathcal{X}(\Gamma), \mathcal{C}(V)$$

$$\mathsf{supp}(\mathsf{F}(\mathsf{f})) \subset \{x \in V : d(x, \mathsf{supp}(\mathsf{f})) \leq n\}$$

Discrete Riemannian Manifolds

$\triangleright$ ν, μ positive dense measures on V

$\triangleright$ B field of symmetric definite positive bilinear forms

$\triangleright$ Riemannian structure $(\mathsf{B}, \mu) \Rightarrow \mathsf{M}, \mathsf{A} = \mathsf{M}^{-1}$
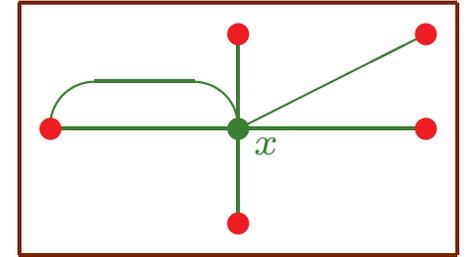
$\star$ symmetric and definite positive

$\triangleright$ Inner product of functions

$$\int_V uv \, d\nu = \sum_{x \in V} u(x) \, v(x) \, \nu(x), \quad u, v \in \mathcal{C}(V)$$

$\triangleright$ Inner product of fields

$$\frac{1}{2} \int_V \mathsf{B}(\mathsf{f}, \mathsf{g}) \, d\mu = \frac{1}{2} \sum_{x \in V} \langle \mathsf{M}(x)\mathsf{f}(x), \mathsf{g}(x) \rangle \, \mu(x), \quad \mathsf{f}, \mathsf{g} \in \mathcal{X}(\Gamma)$$

Difference Operators

▷ Derivative of $u \in \mathcal{C}(V)$: $\qquad \mathsf{d}u = -2u^a$

$$\mathsf{d}u(x) = \sum_{y \sim x} \sum_{e \in E_{xy}} \left(u(y) - u(x)\right) e$$

▷ Gradient of $u \in \mathcal{C}(V)$: $\qquad \nabla u = \mathsf{A}\,\mathsf{d}u$

$$\nabla u(x) = \sum_{e \in E_x} \left[\sum_{y \sim x} \left(u(y) - u(x)\right) \sum_{e' \in E_{xy}} a(x, e, e') \right] e$$

▷ Divergence of $\mathsf{f} \in \mathcal{X}(\Gamma)$: $\qquad \mathsf{div} = -\nabla^*$

$$\mathsf{div}\,\mathsf{f}(x) = \frac{1}{\nu(x)} \sum_{e \in E_x} (\mu f)^a(x, e)$$

▷ Curl of $\mathsf{f} \in \mathcal{X}(\Gamma)$: $\qquad \mathsf{curl}\,\mathsf{f} = \frac{1}{\mu}\left(\mathsf{M}\mathsf{f}\right)^s$

$$\mathsf{curl}^* = \mathsf{curl}, \quad \mathsf{div} \circ \mathsf{curl} = 0, \quad \mathsf{curl} \circ \nabla = 0$$

Laplace-Beltrami Operator

▷ Laplacian of $u \in \mathcal{C}(V)$: $\qquad \Delta u = \mathsf{div}\,(\nabla u) = \mathsf{div}\,(\mathsf{A}\,\mathsf{d}u)$
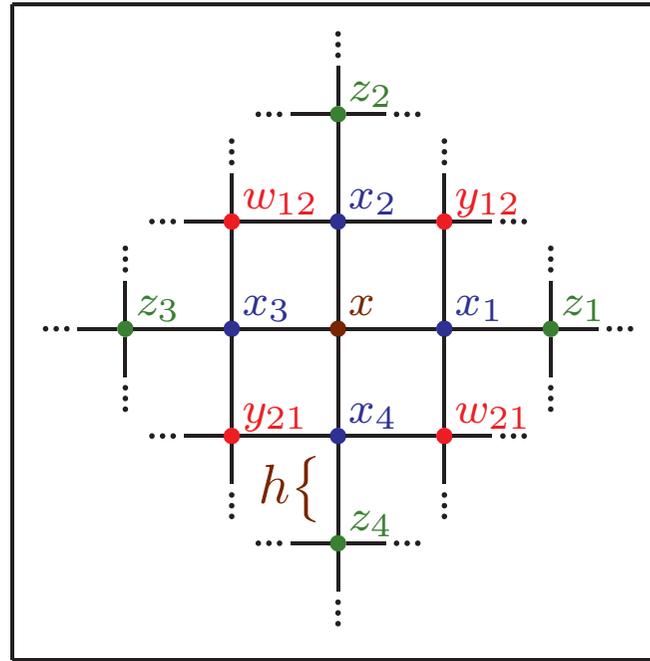
$$\Delta u(x) = \frac{1}{\nu(x)} \sum_{y \in V} c(x,y)(u(y) - u(x))$$

$$c(x,y) = \frac{1}{2} \sum_{z \in V} \Big[\sum_{e \in E_{xy}} \sum_{e' \in E_{xz}} a(x,e,e')\mu(x) + \sum_{e \in E_{yx}} \sum_{e' \in E_{yz}} a(y,e,e')\mu(y) \Big]$$

$$- \frac{1}{2} \sum_{z \in V} \Big[\sum_{e \in E_{zx}} \sum_{e' \in E_{zy}} a(z,e,e')\mu(z) \Big], \quad x \neq y$$

⋆ $-\Delta$ is self-adjoint, of order 2 and positive semidefinite

$\Delta u = 0$ iff $u = $ cte in each connected component of Γ
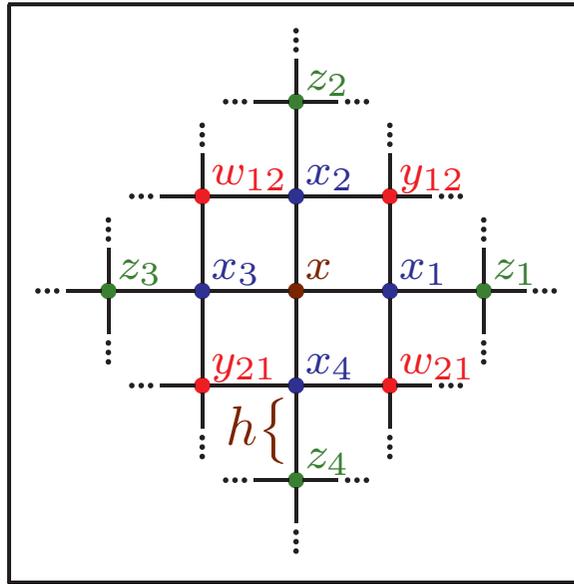
Laplace-Beltrami Operator: Example

$$A = \frac{1}{6h} \begin{pmatrix} 7 & 0 & 1 & 0 \\ 0 & 7 & 0 & 1 \\ 1 & 0 & 7 & 0 \\ 0 & 1 & 0 & 7 \end{pmatrix}$$

$$\implies c(x, x_1) = \frac{4}{3h^2}, \ c(x, z_1) = -\frac{1}{12h^2}, c(x, w_{12}) = 0$$

$\triangleright$

$$-\Delta(u) = \frac{4}{3h^2} \sum_{j=1}^{2} \Big(2u(x) - u(x_j) - u(x_{n+j}) \Big)$$

$$- \frac{1}{12h^2} \sum_{j=1}^{2} \Big(2u(x) - u(z_j) - u(z_{n+j}) \Big)$$

Laplace-Beltrami Operator: Example

$$A = \frac{1}{h} \begin{pmatrix} k_{11} & k_{12} & 0 & 0 \\ k_{12} & k_{22} & 0 & 0 \\ 0 & 0 & k_{11} & k_{12} \\ 0 & 0 & k_{12} & k_{22} \end{pmatrix}$$

$$\implies c(x, x_1) = \frac{k_{11} + k_{12}}{h^2}, \ c(x, z_1) = c(x, y_{12}) = 0, \ c(x, w_{12}) = -k_{12}$$

▷ Standard nine points difference scheme

$$\Delta(u) = \frac{1}{h^2} \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} k_{ij} \Big) \Big(2u(x) - u(x_j) - u(x_{n+j}) \Big)$$

$$- \frac{1}{h^2} \sum_{1 \le i < j \le n} k_{ij} \Big(2u(x) - u(w_{ij}) - u(w_{ji}) \Big)$$

Cohomology of a Discrete Riemannian Manifold

▷ De Rham Complex:

$$\nabla_0 = \nabla, \ \nabla_{2j+1}f = A(Mf)^s, \ \nabla_{2j}f = A(Mf)^a$$

$$\{0\} \xrightarrow{0} \mathcal{C}(V) \xrightarrow{\nabla_0} \mathcal{X}(\Gamma) \xrightarrow{\nabla_1} \mathcal{X}(\Gamma) \cdots \xrightarrow{\nabla_n} \mathcal{X}(\Gamma) \xrightarrow{\nabla_{n+1}} \cdots$$

$$\nabla_n \circ \nabla_{n-1} = 0$$

▷ n-th De Rham cohomology group: $H^n(\Gamma) = \ker\nabla_n/\mathrm{Img}\nabla_{n-1}$

▷ n-th Betti number: $\beta_n = \dim H^n(\Gamma)$

⋆ $\beta_0 = m, \ \beta_1 = |E| - |V| + m, \ \beta_n = 0, \ n \geq 2$

$\implies \chi(\Gamma) = \sum\limits_{n=0}^{\infty}(-1)^n\beta_n$ Euler-Poincaré formula

Cohomology of a Discrete Riemannian Manifold

$\triangleright$ $\delta_n = \nabla_n^* \implies \delta_0 = -\mathsf{div}\,, \;\; \delta_{2j+1}\mathsf{f} = \frac{1}{\mu}(\mu\mathsf{f})^s, \;\; \delta_{2j} = \frac{1}{\mu}(\mu\mathsf{f})^a$

$$\{0\} \xleftarrow{\;0\;} \mathcal{C}(V) \xleftarrow{\;\delta_0\;} \mathcal{X}(\Gamma) \xleftarrow{\;\delta_1\;} \mathcal{X}(\Gamma) \cdots \xleftarrow{\;\delta_n\;} \mathcal{X}(\Gamma) \xleftarrow{\;\delta_{n+1}\;} \cdots$$

$$\delta_{n-1} \circ \delta_n = 0$$

$\triangleright$ Hodge's Laplacian

$$\Delta_n = \delta_n \circ \nabla_n + \nabla_{n-1} \circ \delta_{n-1}$$

$\star$ Hodge's decomposition

$$\mathcal{X}(\Gamma) = \ker\Delta_n \oplus \mathrm{Img}\nabla_{n-1} \oplus \mathrm{Img}\delta_n$$

$$H^n(\Gamma) \simeq \ker\Delta_n \implies H^1(\Gamma) \simeq \{\mathsf{f} \in \mathcal{X}(\Gamma) : \mathsf{div}\,\mathsf{f} = 0, \;\; \mathsf{curl}\,\mathsf{f} = 0\}$$

Cohomology of a Discrete Riemannian Manifold

▷ Hodge decomposition $n = 1$

For each $\mathsf{f} \in \mathcal{X}(\Gamma)$, $\exists \ u \in \mathcal{C}(V)$, $\mathsf{g}, \mathsf{h} \in \mathcal{X}(\Gamma)$:

$$\mathsf{f} = \nabla u + \mathsf{curl}\,(\mathsf{g}) + \mathsf{h}$$

where $\mathsf{div}\,(\mathsf{h}) = 0$ and $\mathsf{curl}\,(\mathsf{h}) = 0$

▷ B orthogonal and compatible and $\mu = \lambda$

$$\{0\} \xrightarrow{\ 0\ } \mathcal{C}(V) \xrightarrow{\ \hat{\mathsf{d}}\ } \mathcal{C}(E) \xrightarrow{\ 0\ } \{0\}$$

$$\{0\} \xrightarrow{\ 0\ } \mathcal{C}(V) \xrightarrow{\ \nabla\ } \mathcal{X}^a(\Gamma) \xrightarrow{\ \nabla_1\ } \{0\}$$

▷ $\mathsf{curl}\,(\mathsf{f}) = \mathsf{M}\mathsf{f}^s \implies \ker(\mathsf{curl}\,) = \mathcal{X}^a(\Gamma)$, $\mathsf{M}^{-2}\mathsf{curl}^2 = \Delta_1 + \nabla \circ \mathsf{div}$

▷ $\Delta_2 = \Delta_3 = \ldots = I$